FEATURE

Control of Constrained Nonlinear Systems: A Case Study

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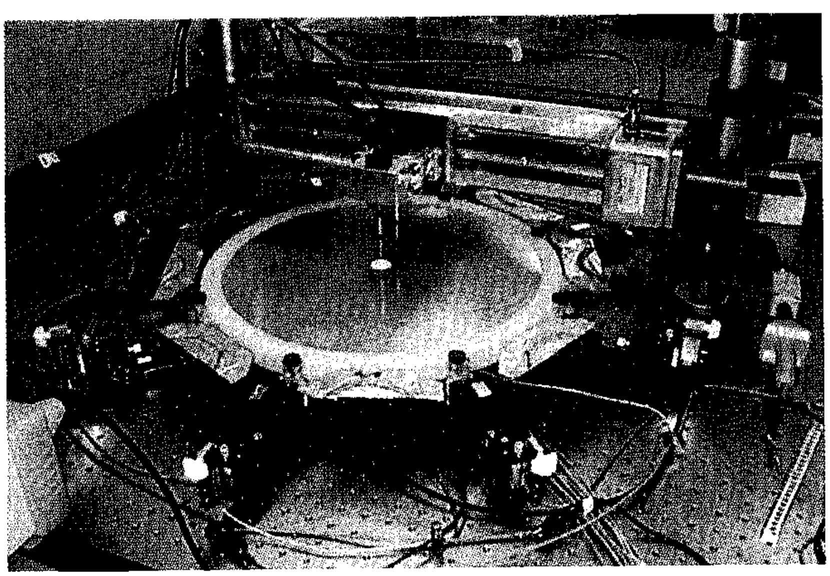

THIS ARTICLE PRESENTS A Lyapunov function approach to the control of nonlinear systems that are subject to pointwise-in-time constraints on state and control. This approach is applied to an electromechanical system that serves as a prototype for the first mode of an electrostatically shaped membrane.

Electrostatically shaped membranes have been proposed as mirrors and antennas since the early 1960s [1]-[4] because they can be used as lightweight reflectors for radar, radio, and optics applications. Lightweight reflectors are in demand, for example, in spacecraft applications where launch weight is a significant constraint. A thin, electrically conducting membrane is formed into a desired shape by

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electrostatic forces that are controlled by varying the electrical potential between the membrane and an electrode mounted below it. Because the membrane is under lateral in-plane tension and a uniform normal stress due to the electrostatic potential, it assumes a paraboloidal shape for optics applications. Since the focal length can be varied by changing the gap distance, electrostatically controlled membranes are particularly suitable for adaptive optics applications [5]. Small focal lengths needed for many applications can be achieved if the gap distance between the membrane and the fixed plate is made sufficiently small. Also, the voltage required to maintain the equilibria with a small gap distance decreases as the gap distance decreases [4]. Consequently, steady-state operation at equilibria with small gap distance values is highly desirable. However, these equilibria are open-loop unstable [4], and active control is needed to achieve their stabilization for a suitable range of gap set points.

The highly nonlinear behavior of the electromagnetically actuated mass-spring damper suggests the use of Lyapunov function methods.

Unfortunately, this technology was essentially abandoned due to difficulties in stabilizing the membrane at desired open-loop unstable equilibria. Thus, it is an application where control may be a critical enabling technology. P.D. Washabaugh has developed an experimental testbed for studying the application of advanced membrane control algorithms (see photograph on previous page). While the deflection of the membrane is described by a partial differential equation, the control voltage is scalar. The laboratory membrane exhibits the open-loop unstable equilibria and is subject to severe state and control constraints. The constraints are due to the limits on the maximal voltage the amplifier can deliver, membrane collisions with the fixed electrode, and electric field breakdown.

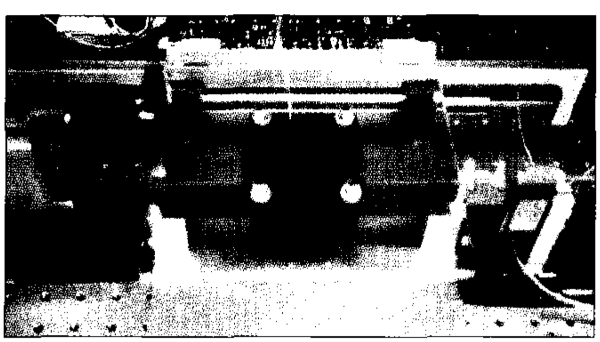

Figure 1. *The electromagnetically actuated mass-spring damper.*

Concurrent with the membrane work, experiments on a simpler physical system that illustrates similar control difficulties were deemed important. This led to the system considered in this article: an electromagnetically actuated mass-spring damper (EAMSD) that exhibits the same general behavior as the first mode of an electrostatically controlled membrane [6]-[8]. It has been implemented in the control laboratory of D.S. Bernstein (see Fig. 1). Results presented here include the description of an effective nonlinear control scheme and experiments involving its use with the laboratory hardware. Naturally, the objective of this article falls short of developing a successful control scheme for electrostatically controlled membranes; however, light is shed on the basic control scheme and its effectiveness in dealing with the key problems of electrostatic membrane control.

The highly nonlinear behavior of the EAMSD suggests the use of Lyapunov function methods. Although these methods are popular for designing nonlinear controllers, their application to systems with general reference commands and demanding state and control constraints is problematic. The approach described here takes its inspiration from prior work on reference governors [9]-[15]. Reference governors are auxilary nonlinear systems that filter reference commands to closed-loop systems in such a way that constraints on their internal variables are satisfied. Unlike input preplanning schemes, such as those used in robotics applications to generate constraint-admissible motions, reference governors operate online, responding immediately to reference commands as they occur. Maximal constraint-admissible positively invariant sets play a dominant role in the prior literature.

Our approach exploits a family of Lyapunov functions that are parametrized by set-point-determined equilibria of the closed-loop system. The resulting reference governor generates a piecewise constant output that, subject to future constraint satisfaction, tracks as closely as possible the reference command. The design process has considerable flexibility and is conceptually simple. It should be emphasized that the exploitation of parametrized equilibria is distinctly different than in gain-scheduling of linear designs. In the reference governor approach, it is assumed a priori that a control scheme (perhaps a gain-scheduled one) has been devised that provides good performance in a neighborhood of each set-point determined equilibrium. The central purpose of the reference governor is to guarantee constraint satisfaction in the presence of a very general class of input commands.

This article is organized as follows. The next section describes the nonlinear control algorithm in general terms. Then an empirically based model for the EAMSD is developed that includes inequality constraints on magnet current and mass displacement. Equilibria for the model are then characterized, after which equations that implement the control algorithm for the EAMSD are considered. If the current constraint is sufficiently binding, the set of achievable equilibria is disconnected. Then, as described in the next section, the overall control algorithm must be augmented to include a temporary, unstable phase for implementing transfer between equilibria that belong to separate components of the set of achievable equilibria. Finally, some experiments with the actual hardware are described. The conclusion summarizes the principal features of the reference governor scheme.

Basic Control Algorithms

Suppose the plant model is of the form

$$\dot{x} = F(x, u), \tag{1}$$

where x is an n-vector and u is an m-vector. Our reference governor approach is based on parametrized equilibria of this system that are generated by means of a stabilizing and performance-enhancing controller $u = U(x, r)$. The resulting closed-loop system is

$$\dot{x} = f(x, r) = F(x, U(x, r)), \tag{2}$$

where r is an m-vector reference command which, when held constant, becomes a set point. Set points are restricted to a set S which determines the desired equilibria (i.e., $r \in S$ implies $f(x_e(r), r) = 0$). Note that for some $r \in S$, there may be several equilibria; the functional notation $x_e(r)$ means that one of them has been chosen. A variety of effective design tools exist for determining the controller, $U(x, r)$, at each equilibrium point; see [16] for a general discussion of Control Lyapunov Function (CLF)-based design procedures. Often such controllers can be shown to be optimal with respect to certain types of integral performance criteria and to have an associated Lyapunov function. Dynamic controllers can also be introduced. In this case, the state in (2) is augmented to include controller states.

The key to our approach is an r-parametrized Lyapunov function for (2), $V(x, r)$. Specifically, we suppose that $V(x, r)$ is smooth and for each $r \in S$ satisfies the following assumptions: (a) $V(x, r) \geq 0$; (b) $V(x, r) = 0$ only for $x = x_e(r)$; (c) there is a $q(r) > 0$ such that the sublevel set

$$\Pi(r) = \{x : V(x, r) \leq q(r)\} \tag{3}$$

is bounded; (d) for all $x \in \Pi(r)$

$$\dot{V}(x, r) = \frac{\partial V}{\partial x}(x, r) f(x, r) \leq 0. \tag{4}$$

Then, under an additional LaSalle invariance assumption [17], it follows that $\Pi(r)$ is a positively invariant set that is a

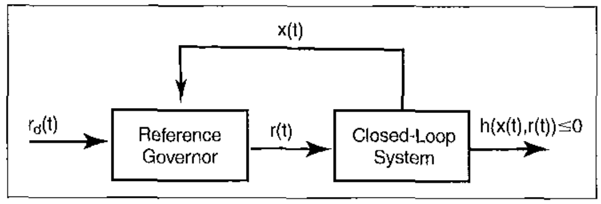

Figure 2. *Reference governor: The arrangement of the control algorithm.*

domain of attraction for $x_e(r)$ [i.e., $x(0) \in \Pi(r)$ implies $x(t) \in \Pi(r)$ all $t > 0$ and $x(t) \to x_e(r)$ as $t \to \infty$]. Suitable Lyapunov functions are often associated with controller synthesis methods, or they can be generated by recursive procedures [16]. Another approach is to feedback linearize (1) by $U(x, r)$. $V(x, r)$ can then be generated by solving a Lyapunov equation associated with the resulting linear system (2). See [18] and [19] for examples of this approach in the context of automotive control problems.

In most applications, such as the EAMSD, state and control constraints are imposed by physical limitations of the hardware or safety considerations. Usually these constraints can be represented by a system of nonlinear inequalities:

$$H(x, u) \leq 0, \tag{5}$$

where $H : \mathrm{R}^{n+m} \to \mathrm{R}^p$. For the closed-loop system, these state and control constraints take the form of parametrized state constraints:

$$h(x, r) = H(x, U(x, r)) \leq 0. \tag{6}$$

In general, it is difficult to handle these constraints directly in the context of Lyapunov-based designs so that for all set points $r \in S$, $U(x, r)$ provides acceptable operation of the closed-loop system.

To see why this is so, consider first the obvious Lyapunov-function-based approach to constraint enforcement. Make the following additional assumption: (e) for each $r \in S$, $q(r) > 0$ and

$$x \in \Pi(r) \Rightarrow h(x, r) \leq 0. \tag{7}$$

Then $\Pi(r)$ is a safety domain for the equilibrium $x_e(r)$. Indeed, if $x(0) \in \Pi(r)$, then the trajectory of the closed-loop system satisfies the constraint: $h(x(t), r) \leq 0$ for all t. Clearly, it is advantageous to find the largest $q(r)$ that causes assumption (e) to be satisfied. Then $\Pi(r)$ is the largest set provided by a given V that is safe under the constraints.

Unfortunately, in many applications, even the maximal set $\Pi(r)$ may be fairly small. This creates problems for the transient operation of the closed-loop system caused by large set-point changes. To be more specific, let the desired reference command $r_d(t)$ change at a time instant

$t = 0$ from a value r_i to the value r_f and let $r_d(t) = r_f$ for $t > 0$. Suppose, furthermore, that the system is initially at the equilibrium $x(0) = x_e(r_i)$, $h(x_e(r_i), r_i) \le 0$. If $|r_f - r_i|$ is large, it is likely that $x(0) \notin \Pi(r_f)$; then the application of the feedback controller $U(x, r_f)$ may not result in the safe operation of the closed-loop system.

Our approach addresses such limitations of Lyapunov-function designs by modifying how changes in the reference commands are generated. For simplicity, let us suppose for the present that $r_d(t)$ is a step change from r_i to r_f as described in the preceding paragraph. Later, it will become apparent that the reference governor functions well for more general reference commands. The idea is to replace the actual reference command r_f by $r(t) \equiv r_0$, which is as close as possible to $r_d(0) = r_f$ but satisfies $x(0) \in \Pi(r_0)$. The latter inclusion guarantees safety if $r(t) = r_0$ for all $t \ge 0$, but the trajectory will not converge to the desired equilibrium $x_e(r_f)$. Hence, an attempt to modify the reference command $r(t)$ is made at the next sampling instant T, where we again seek a reference command value r_1 that is as close as possible to $r_d(T) = r_f$, subject to the constraint that $x(T) \in \Pi(r_1)$. Since the reference command, $r(t)$, is kept constant in the intervals $0 \le t < T$ and $T \le t < 2T$, the Lyapunov theory applies in each interval. The procedure is continued on subsequent intervals $kT \le t < (k+1)T$, where k is an integer. Since $x(t) \in \Pi(r_k)$ for all $t \in [kT, (k+1)T]$, it follows that the constraint, $h(x(t), r(t)) \le 0$, is satisfied for all $t \ge 0$. Fig. 2 illustrates this process of modifying the reference command from $r_d(t)$ to $r(t)$ as required to meet the pointwise-in-time constraints.

What is new is the emphasis on implementation of a reference governor in the context of general Lyapunov-based designs for continuous-time nonlinear systems.

The explicit process by which the r_k are determined is computationally straightforward. We want $r(t)$ to move toward $r_d(t) = r_f$ from its current value. This requirement is implemented by setting, for $k = 0, 1, \ldots$,

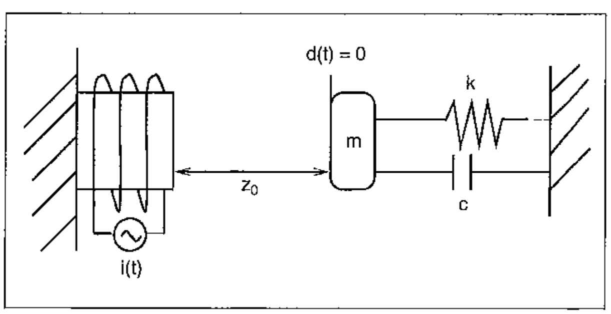

Figure 3. *Notation for the EAMSD.*

$$r_k = r_{k-1} + \lambda_k(r_d(kT) - r_{k-1}), \tag{8}$$

where $r_{-1} = r_i$ and λ_k solves the optimization problem of maximizing λ subject to $0 \le \lambda \le 1$ and $x(kT) \in \Pi(r_{k-1} + \lambda(r_f - r_{k-1}))$. Equivalently, λ_k solves the problem:

$$\text{maximize } \lambda \text{ subject to}$$

$$0 \le \lambda \le 1,\ \mathcal{F}(x(kT), r_{k-1} + \lambda(r_d(kT) - r_{k-1})) \le 0\,, \tag{9}$$

where

$$\mathcal{F}(x, r) = V(x, r) - q(r). \tag{10}$$

Since λ is a scalar, the optimization problem can be solved quickly by a search procedure.

A theoretical justification of the control scheme is given in [20]. In addition to assumptions (a)-(e), three more are required. While technical in character, they are intuitively reasonable. Roughly speaking, they have the following objectives: (f) restricts r so that $F(x(kT), r_{k-1} + \lambda(r_d(kT) - r_{k-1}))$ is defined for all $\lambda \in [0,1]$; (g) establishes in a precise way the required smoothness of the functions f, V, and q; (h) implies there is a decrease in F for each reference governor time step. Specifically, the assumptions are: (f) S is compact and convex; (g) $f(x, r)$ and $V(x, r)$ are continuously differentiable on R^{n+m}, and $q(r)$ is continuous on S; (h) for every fixed $r \in S$, $V(x, r)$ satisfies a LaSalle-like condition in [20] on $\Pi(r)$. The verification of conditions (f)-(h) for the EAMSD is discussed later. There, as in most practical situations, the conditions occur naturally as a consequence of constructing a $V(x, r)$ that satisfies assumptions (a)-(e).

Under assumptions (a)-(h), the reference governor is guaranteed [20] to behave nicely. It is not necessary to assume that $x(0) = x_e(r_i)$ or that $r_d(t) = r_f$ for all $t \ge 0$. Suppose that $r_d(t) \in S$ for $t \ge 0$ and that

$$x(0) \in X = \bigcup_{r \in S} \Pi(r)\,. \tag{11}$$

Choose an r_{-1} so that $x(0) \in \Pi(r_{-1})$. Then, from the above arguments and (9), it follows that the r_k belong to S and are well defined. Moreover, $x(t) \in \Pi(r_k)$ for $t \in [kT, (k+1)T]$, $k \ge 0$. Thus, the constraints $h(x(t), r(t)) \le 0$, $t \ge 0$, are satisfied. Suppose further that there is a $\bar{t} \ge 0$ such that $r_d(t) = r_f \in S$ for all $t \ge \bar{t}$. Then [20] shows that the r_k have a finite settling time (there is an integer $\bar{k} \ge 0$ such that $r(t) = r_f$ for all $t \ge \bar{k}T$) and $x(t) \to x_e(r_f)$ as $t \to \infty$.

Note that the condition (11) does not depend on r_f and that the set X may be quite large. Thus, by choosing r_{-1} so that $x(0) \in \Pi(r_{-1})$, it is possible to extend significantly the range of initial conditions that may be handled. There are several practical ways for finding r_{-1}. It is only necessary that $r_{-1} \in S$ and $\mathcal{F}(x(0), r_{-1}) \le 0$. One approach is to choose a finite grid of points $\tilde{r}_j \in S$, $j \in J$, such that $\tilde{X} = \bigcup_{j \in J} \Pi(\tilde{r}_j) \subset X$ approximates X. Then, for $x(0) \in \tilde{X}$, r_{-1} can be found by evaluating $\mathcal{F}(x(0), \tilde{r}_j)$ for $j \in J$ until $\mathcal{F}(x(0), \tilde{r}_j) \le 0$. Another approach is to apply an iterative procedure for the minimization of $\mathcal{F}(x(0), r)$ for $r \in S$. In general, it is not necessary to obtain the minimum; the iterative process is terminated when $\mathcal{F}(x(0), r) \le 0$.

The general idea of the control algorithm is not new and, in fact, underlies much of the research on reference governors (see, e.g., [9]-[15]) and, more generally, the use of positively invariant sets for constraint satisfaction (see, e.g., [21] for an extensive literature review). See also [22], which exploits nested families of Lyapunov functions to develop control laws for robots moving in an obstacle-restricted space. What is new is the emphasis on implementation of a reference governor in the context of general Lyapunov-based designs for continuous-time nonlinear systems.

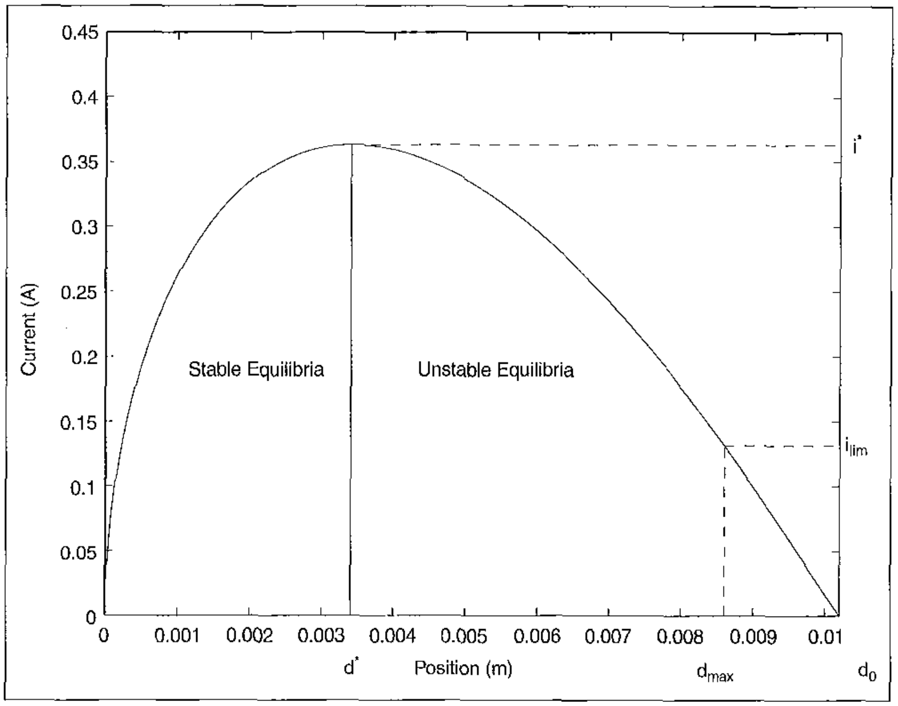

Figure 4. *Equilibria positions.*

Model for the EAMSD

Fig. 3 illustrates the basic features of the EAMSD. Idealized equations of motion are

$$\ddot{d}(t) = -\frac{k}{m}d(t) - \frac{c}{m}\dot{d}(t) + \frac{\alpha}{m}\frac{i(t)^2}{(z_0 - d(t))^2}, \tag{12}$$

where m is the mass, k is the spring constant, c is the damping constant, $i(t)$ is the current in the electromagnet coil, $d(t)$ is the position of the mass, z_0 is the distance between the electromagnet and the mass when $i = 0$, and α is a current-to-force constant. Experimental measurements lead to a somewhat different form for the last term. Specifically, in MKS and ampere units,

$$\ddot{d} = -\frac{k}{m}d - \frac{c}{m}\dot{d} + \frac{\alpha}{m}\frac{i(t)^\beta}{(d_0 - x)^\gamma}, \tag{13}$$

where $\alpha = 4.5 \times 10^{-5}$, $\beta = 1.92$, $\gamma = 1.99$, $c = 0.0659$, $k = 38.94$, $z_0 = 0.0086$, $d_0 = 0.0102$, and $m = 1.54$.

Reformulating the equations of motion into the form (1) gives

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\alpha}{m}\frac{1}{(d_0 - x_1)^\gamma}\end{bmatrix}u, \tag{14}$$

where $d = x_1$, $\dot{d} = x_2$, and the control input is $u = i^\beta$.

The control is constrained to be nonnegative,

$$u(t) \ge 0. \tag{15}$$

This is a pointwise-in-time control constraint that renders the system underactuated in a physical sense. Specifically, the control can pull the mass closer to the electromagnet coil, but it cannot push the mass away from the electromagnet coil. The maximal current is also limited, thereby leading to another pointwise-in-time control constraint,

$$u(t) \le u_{\max} = (i_{\max})^\beta. \tag{16}$$

Collisions of the mass with the fixed electromagnet must be avoided. Thus, $d(t) \le z_0$. In fact, it may be wise to provide a margin of safety or limit peak swings of transient responses. For this reason, we introduce $d_{\max} < z_0$ and require

$$d(t) \le d_{\max}. \tag{17}$$

Open-Loop Equilibria

For constant u, the equilibria are obtained by equating the right-hand side of (14) to zero. For $x_1 = d_{eq}$, $x_2 = 0$, and $u(t) = \bar{u}$, this gives

$$\frac{kd_{eq}(d_0 - d_{eq})^\gamma}{\alpha} = i^\beta = \bar{u}. \tag{18}$$

The resulting relationship between d_{eq} and i is shown in Fig. 4. The maximum current allowed by the relationship is $i^* = 0.3631$. Denote the corresponding values of $\bar{u}$ and d by $u^* = (i^*)^\beta = 0.1430$ and $d^* = 0.0034$. For $0 \le \bar{u} < u^*$, there are two equilibria for d: $0 \le d^1_{eq}(\bar{u}) < d^*$ and $d^* d^2_{eq}(\bar{u}) \le d_0$; for $\bar{u} = u^*$, there is one equilibrium value: $\hat{d} = d^*$. Thus, for $u_{\max} \ge u^*$, the set of constraint-admissible steady state displacements, d_{eq}, is the interval $[0, d_{\max}]$. Let $u_{\lim}$ be determined by $d^2_{eq}(u_{\lim}) = d_{\max}$. Then for $u_{\lim} < u_{\max} < u^*$, the set of steady-state constraint-admissible displacements is the union of two disjoint intervals, $[0, d^1_{eq}(u_{\max})] \cup [d^2_{eq}(u_{\max}), d_{\max}]$; for $0 < u_{\max} < u_{\lim}$, it is the interval $[0, d^1_{eq}(u_{\max})]$.

To investigate local stability of constraint-admissible equilibria, we linearize (14) at $x_1 = d_{eq}, x_2 = 0$, and $u = \bar{u}$ and obtain

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} + \frac{\gamma\alpha}{m}\frac{\bar{u}}{(d_0 - d_{eq})^{\gamma+1}} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\alpha}{m}\frac{1}{(d_0 - d_{eq})^\gamma} \end{bmatrix} \delta u. \tag{19}$$

The necessity of dealing with multiple modes of the membrane is essential to a successful design of an experimental control scheme for this more complex application.

Since $c > 0$, we understand that the necessary and sufficient condition for an equilibrium point to be locally asymptotically stable is that the stiffness coefficient, K, is positive:

$$K = \frac{k}{m} - \frac{\gamma\alpha}{m}\frac{\bar{u}}{(d_0 - d_{eq})^{\gamma+1}} > 0. \tag{20}$$

It is easy to confirm that constraint-admissible equilibria are stable for $0 \le d_{eq} < d^*$ and unstable for $d^* \le d_{eq} \le d_{\max}$.

To summarize, we have demonstrated that for $u_{\max} > u_{\lim}$, the set of constraint-admissible open-loop equilibria splits into two intervals; a stable interval and an unstable interval. If the control current constaint is tight ($u_{\max} < u^*$), these intervals are disconnected.

Implementation of the Reference Governor

We now apply the ideas discussed under "Basic Control Algorithms" to the EAMSD. The main steps are the choice of $U(x, r)$ and $V(x, r)$, application of the constraint conditions to the determination of $q(r)$, verification of assumptions (a)-(h), and definition of the control equations.

We choose $U(x, r)$ so that it stabilizes the equilibria and linearizes the resulting closed-loop system:

$$U(x, r) = \alpha^{-1}(d_0 - x_1)^\gamma (kr - c_d x_2). \tag{21}$$

Then (2) becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k}{m}(x_1 - r) - \frac{c + c_d}{m} x_2, \tag{22}$$

a mass-spring damper whose equilibrium state is $x_e(r) = (r, 0)$. While the controller allows an increase in system damping, it is "low gain" in the sense that the natural frequency of the open-loop system is not increased by feedback.

For simplicity, we consider first the case where $c_d = 0$ and $i_{\max}$ is significantly larger than i^*. This eliminates constraint (16). Constraint (15) together with (21) requires $r \ge 0$, a condition that is satisfied by choosing S so that it contains no negative elements. The remaining constraint, (17), gives $h(x, r) = h_1(x, r) = x_1 - d_{\max}$. From $h(x_e(r), r) \le 0$, it follows that constraint-admissible set points must satisfy $r \le d_{\max}$. Thus, $S \subset [0, d_{\max}]$.

A natural choice for $V(x, r)$ is the "energy" of the system (22) relative to $x_e(r)$:

$$V(x, r) = \frac{m}{2}x_2^2 + \frac{k}{2}(x_1 - r)^2. \tag{23}$$

Since $\dot{V} = -(c + c_d)x_2^2$, it follows that assumptions (a)-(d) are satisfied.

To find the most effective $q(r)$ that satisfies (e), we maximize $q(r)$ subject to condition (7). This optimization problem has a simple geometric interpretation: $\Pi(r)$ touches the set $\{x : h_1(x, r) \ge 0\}$, but there is no $x \in \Pi(r)$ such that $h_1(x, r) > 0$. Thus,

$$q(r) = \min_x V(x, r) \text{ subject to } h_1(x, r) = 0. \tag{24}$$

It follows that

$$q(r) = \frac{k}{2}(d_{\max} - r)^2. \tag{25}$$

Clearly, (e) is satisfied if r is restricted so that $q(r) > 0$. This is where assumption (f) comes in; S must be a closed interval, chosen so that it excludes $r = d_{\max}$. The obvious choice for S is $S = [0, d_{\max} - \varepsilon]$ where $\varepsilon > 0$. Since (f) and (g) are now satisfied, it remains to verify (h). This condition [20] requires that for all constant $r \in S$, the solutions of (22), starting in the set $\{x : 0 < V(x,r) \le q(r), \dot{V}(x,r) = 0\}$ immediately leave the set. The property is easily confirmed.

To complete the statement of the control law, it is necessary to solve the λ-optimization problem (9). Normally, the solution must be obtained by an iterative procedure. However, in this case, $\mathcal{F}(x, r)$ is linear in r and there is a formula for λ: if $\mathcal{F}(x(kT), r_d(kT)) \le 1$, then $\lambda = 1$; otherwise,

$$\lambda = (\rho(x(kT)) - r_{k-1})(r_d(kT) - r_{k-1})^{-1}, \quad (26)$$

where

$$\rho(x) = \frac{1}{2}\left(d_{\max}^2 - x_1^2 - \frac{m}{k}x_2^2\right)(d_{\max} - x_1)^{-1}. \quad (27)$$

When $i_{\max} < i^*$ and $c_d \ne 0$, the derivation of the control equations becomes more complex. Then, all three of the constraints (15)-(17) become active and $h(x, r)$ has three components. Each of them is handled separately in the manner described in the preceding paragraphs. Let the resulting occurrences of q and λ be denoted by $q^i(r)$ and λ^i, $i = 1,2,3$. Then, $q(r) = \min\{q^1(r), q^2(r), q^3(r)\}$ and $\lambda = \min\{\lambda^1, \lambda^2, \lambda^3\}$. Note that while $q(r)$ is no longer differentiable, it satisfies assumption (g) because it is continuous. Similarly, the requirement on S becomes more complex. For example, if $c_d = 0$ it follows that the set of constraint-admissible set points is $E_1 \cup E_2$ where $E_1 = [0, d_{eq}^1(u_{\max})]$ and $E_2 = [d_{eq}^2(u_{\max}), d_{\max}]$. Since the two intervals are disjointed and $S \subset E_1 \cup E_2$ is convex, S is given by either $S_1 = [0, d_{eq}^1(u_{\max}) - \varepsilon]$ or by $S_2 = [d_{eq}^2(u_{\max}) + \varepsilon, d_{\max} - \varepsilon]$. Thus, the reference governor cannot achieve transfers between all equilibria determined by $r \in S_1 \cup S_2$.

A Control Scheme for Disconnected Equilibria

We now discuss, in general terms, "bridging strategies" that allow automatic set-point transfers between disconnected sets such as S_1 and S_2 in the preceding paragraph.

Suppose reference governors have been designed for each of the two intervals and r_{-1} is determined by $x(0) \in \Pi(r_{-1})$, as described in "Basic Control Algorithms." Let X_1 and X_2 denote the corresponding sets defined by (11). Suppose we want to move from a state $x(0) \in X_1$ to $X_e(r_f)$ where $r_f \in S_2$. To achieve this objective, it is only necessary, for some $\rho_1 \in S_1$, to invent a constraint-admissible bridging controller, $U_B(x)$, that causes the state of the resulting closed-loop system to move from initial states in a neighborhood, N, of $x_e(\rho_1)$ to states in X_2. The overall control strategy consists of the following steps: (1) apply the reference governor on S_1 with $r_d(t) = \rho_1$ and continue its operation until there is an integer k_1 such that $x(k_1T) \in N$; (2) starting at $t = k_1T$, apply the bridging controller and increase t until there is a k_2 such that $x(k_2T) \in X_2$; and (3) starting at $t = k_2T$, apply the reference governor on S_2 with r_{-1} determined by

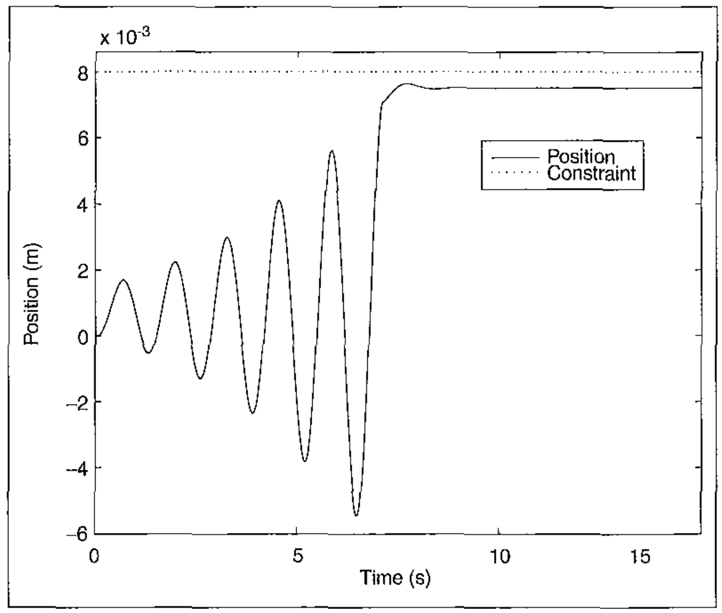

Figure 5. *Bridging controller with reference governor.*

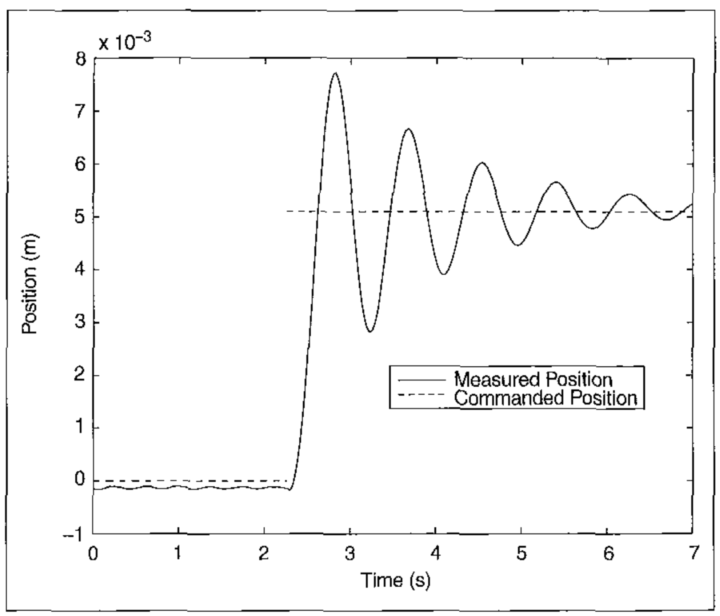

Figure 6. *Position response without reference governor.*

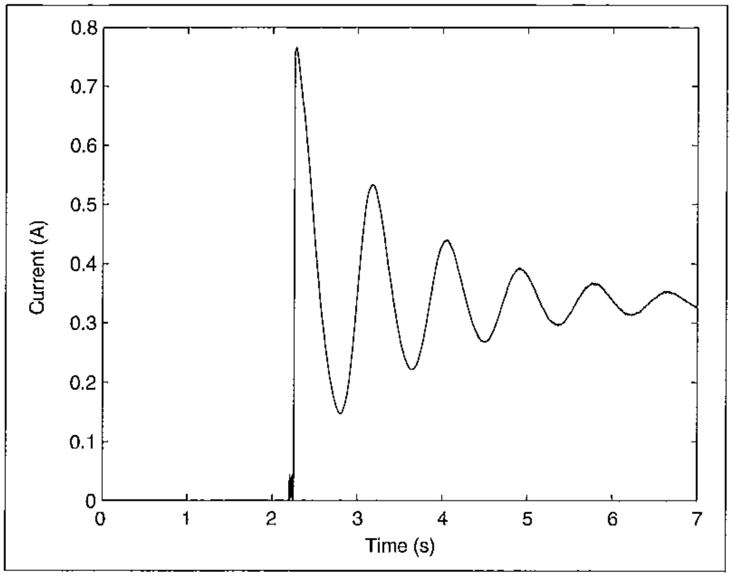

Figure 7. *Current response without reference governor.*

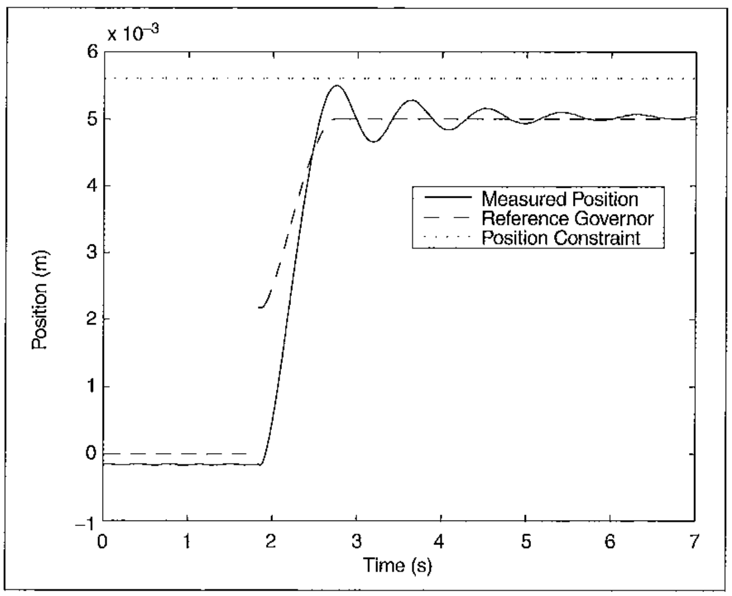

Figure 8. *Position response with reference governor, d_{max} = 0.0056.*

$x(k_2T) \in \Pi(r_{-1})$ and $r_d(t) \equiv r_f$. Clearly, this strategy leads to the set point r_f and then, as $t \to \infty$, $x(t) \to x_e(r_f)$.

Fig. 5 illustrates the simulated response of a bridging controller for the EAMSD. It destabilizes motions for initial conditions in a neighborhood N of $x_e(0) = 0$, leads them into X_2, and satisfies the current constraints. The controller is defined by

$$U_B(x) = \min\{U(x,\delta), u_{max}\} \quad \text{when } x_2 \geq 0$$
$$= 0 \quad \text{when } x_2 < 0. \tag{28}$$

Here, $U(x,\delta)$ is given by (21) with $c_d = -20 < -c$. The parameter δ is a small positive number that guarantees that the motions generated by U_B move away from $x(0) \in N$. Note that $U_B(x) \geq 0$ when $x_2 > 0$. Thus, (28) automatically imposes the constraints (15) and (16). The constraint $x_1(t) \leq d_{max}$ is confirmed directly by simulations. For $x(0) \in N$, it was also confirmed that $x(0)$ eventually enters $\Pi(0.0075) \subset X_2$. Figure 5 illustrates the result. At $t \approx 7$ s, the reference governor for S_2 is engaged with $r_d(t) \equiv \rho_2 = 0.0075$, and it brings $x(t)$ to the corresponding equilibrium $(\rho_2, 0)$. Note that position constraint $x_1(t) \leq 0.008$ is satisfied for all t.

Experimental Results

Ancillary equipment associated with the EAMSD consisted of an Intel-based computer with a DSPACE controller board, a differential-transformer position sensor, and a current amplifier. Position data were sampled at a 1-kHz rate. The mass velocity was estimated using a finite-difference filter based on the bilinear transform and approximating the transfer function $G(s) = s(\tau s + 1)^{-1}$. A break frequency of 20 Hz provided accurate velocity data with a dynamic lag negligible compared to the system response times. Controller code was generated from a Simulink block diagram and cmex file using the Real-Time Workshop in MATLAB. The inherent damping in the mass-spring damper is very small, so extra damping was added by setting $c_d = 3$.

Since the maximum amplifier current exceeds i^* by a wide margin, constraint (16) is inactive, even though $c_d \neq 0$. Thus, λ is given by (26) and r_k by (8). Each full computation of r_k takes approximately 100 μs. It is implicit in the description of the reference governor that r_k is computed instantaneously from $x(kT)$. Practically, this assumption is met if $T >> 10^{-4}$. In fact, it was possible to operate the system successfully with $T = 10^{-3}$, the basic sampling period of the interface.

Figs. 6-9 illustrate some of the experimental results. In both cases considered, the reference command of position moves from $r_d = 0$ (an open-loop stable equilibrium) to $r_d = 0.0051$ (an open-loop unstable equilibrium). Figs. 6 and 7 show the position and current responses for the controller (21) with no reference governor. The system is lightly damped, resulting in a large overshoot in both position and current. In fact, the peak value in the position $d(t)$ is almost 0.0077, which is very close to the mass-magnet collision limit $z_0 = 0.0086$. Figs. 8 and 9 show the responses with the reference governor and the position constraint

$d_{max} = 0.0056$, which is 10% larger than the set point. The peak position almost reaches but does not exceed the constraint. The output of the reference governor has a sizable jump at the time of change in the set-point command, but then increases smoothly for about 1.0 s until it reaches the set-point command. This delay is small compared with the overall settling time; however, the minimum distance between the mass and the magnet is increased from 0.0009 to 0.0031. It is interesting to note that in Fig. 9, the peak current is less than 0.46 A. Thus, the position constraint has also produced an appreciable reduction of the peak current.

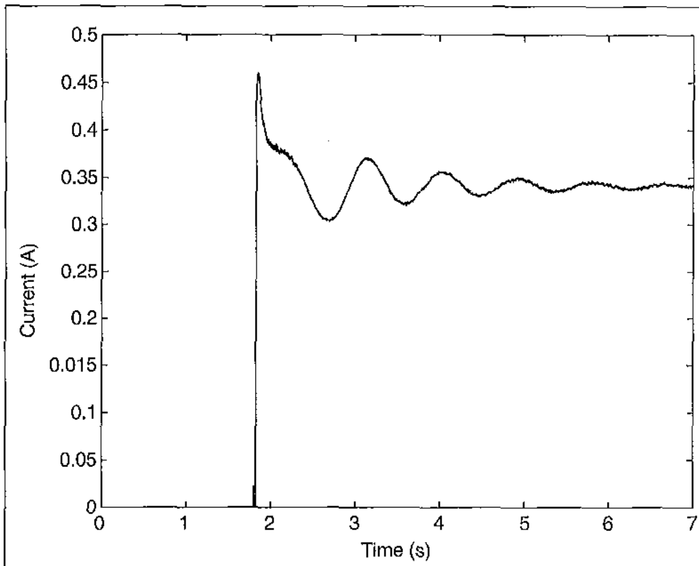

Figure 9. *Current response with reference governor,* $d_{max} = 0.0056$.

Conclusion

This article has described a conceptually simple and practical approach to a difficult class of problems: set-point control of nonlinear systems that are subject to hard constraints on state and control variables. It exploits a family of Lyapunov functions parametrized by the set points and constructed to meet an intuitively sensible requirement: for each set-point-determined equilibrium state, the Lyapunov function defines a constraint-admissible domain of attraction. This property is the basis for a finite settling-time reference governor that filters the set-point commands and generates constraint-admissible motions. Online implementation of the governor is straightforward, requiring at most the solution of several nonlinear root-finding problems in a single variable.

The approaches taken in the prior literature have several advantages over our Lyapunov function approach. Since they are based on maximal constraint-admissible invariant sets, the resulting reference governors respond more rapidly and function over the larger set of initial conditions. Further, for discrete-time linear systems, it is often possible to obtain explicit representations for the invariant sets, thus avoiding the problem of finding an appropriate family of Lyapunov functions. However, they also have disadvantages: the theory is more complex, and it does not apply to continuous-time nonlinear systems.

The purpose of the article has been to describe the details of the approach and illustrate their application using an interesting example, the EAMSD. Laboratory experiments demonstrate the effectiveness of the resulting reference governor. From their success with this and another example application, the authors believe that the described methodology has much to offer.

The EAMSD is a prototype problem for a more complex system, an electrostatically actuated membrane. The necessity of dealing with multiple modes of the membrane is essential to successful design of an experimental control scheme for this more complex application; this design will be pursued in our future work.

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